IFUP-TH/2008-35
October, 2008

On the Stability of Non-Abelian Semi-local Vortices

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Abstract

We study the stability of non-Abelian semi-local vortices based on an $\mathcal{N}=2$ supersymmetric $H = SU(N_\mathrm{c}) \times U(1)/\mathbb{Z}_{N_\mathrm{c}} \sim U(N_\mathrm{c})$ gauge theory with an arbitrary number of flavors ($N_f > N_c$) in the fundamental representation, when certain $\mathcal{N}=1$ mass terms are present, making the vortex solutions no longer BPS-saturated. Local (ANO-like) vortices are found to be stable against fluctuations in the transverse directions. Strong evidence is found that the ANO-like vortices are actually the true minima. In other words, the semi-local moduli, which are present in the BPS limit, disappear in our non-BPS system, leaving the vortex with the orientational moduli $\mathbb{C}P^{N_\mathrm{c}-1}$ only. We discuss the implications of this fact on the system in which the $U(N_\mathrm{c})$ model arises as the low-energy approximation of an underlying e.g. $G = SU(N_\mathrm{c}+1)$ gauge theory.

1 Introduction

The last several years have witnessed a remarkable progress in our understanding of physics of solitons, in particular, that of vortices in non-Abelian gauge theories. These problems are important as they are intimately related to one of the deepest issues of particle physics such as confinement [1]; they may be important in some condensed-matter physics or cosmological problem as well. The progress has been made mainly in the context of supersymmetric gauge theories where many aspects of calculations such as the quantum modification of the potential and symmetry realizations, are under much better control. After the discovery of vortices carrying continuous, non-Abelian moduli in [2, 3], many related studies have been carried out [4, 5, 6, 7, 8, 9, 10]. For good reviews, see Refs. [11, 12, 13, 14]. Some recent developments involve the following issues:

- A detailed study of moduli and the transformation properties of higher winding vortices [15, 16, 17, 18] has been performed. The last of these papers contains basic results on the vortex transformation properties which allow them to be interpreted in a simple group-theoretical language.
- Another development concerns semi-local vortices in systems with larger number of flavors (matter fields) [2, 19, 20]. As it happens in the $U(1)$ Higgs systems with more than one charged scalar field [21, 22], a new kind of moduli emerges; the thickness of the flux tube is no longer a fixed quantity, but corresponds to one of these moduli. This makes the corresponding vortex moduli space very interesting. In particular, a new type of (Seiberg-like) duality has been found, between different models having dual vortex moduli (and sharing the common sigma-model lump limit at very strong couplings) [20].
- A systematic study of non-BPS local vortices has been initiated in Refs. [23, 24] where several kinds of perturbations have been considered. The interactions have a very rich structure, depending both on the distance and relative orientations, a new feature not present in the Abelian counterpart.
- A particularly significant development concerns the extension of the analysis to systems with generic gauge groups [25, 26, 27]. As compared to the $U(N)$ models studied in most papers, the systems based on, e.g. $U(1) \times SO(N)$, $U(1) \times USp(2N)$ are characterized by a larger vacuum degeneracy, even after the "color-flavor locked" vacuum has been chosen to study the solitons.
- The question of dynamical Abelianization has recently been addressed more carefully in Ref. [28]. Only under definite conditions, the non-Abelian moduli fluctuating along the vortex length and in time (which are described by a two-dimensional sigma-model), are absorbed in the monopoles, turn into the dual gauge group, before Abelianizing dynamically. The recent result on the non-Abelian vortices with a product moduli [29] seems to be particularly significant in this context.
- Further important developments have been achieved in the study of BPS non-Abelian vortices in theories with $\mathcal{N} = 1$ supersymmetry (heterotic vortices) [30, 14]. These works generalize the known links between the vortex and the bulk theory to a less supersymmetric case. It is reasonable to think that heterotic vortices play a role in the physics of Seiberg

duality [31]. However, so far, there seems to be no clean statement about this. Some related works can be found in Refs. [32].

- Another direction involves the non-Abelian vortex in the Higgs vacuum of $\mathcal{N} = 1^*$ theory with gauge group $SU(N_c)$, which is a mass deformation of the conformal $\mathcal{N} = 4$ theory. This issue has been first studied for $N_c = 2$ in Ref. [33]. In Ref. [34] the case of larger N_c has been studied, both in the weakly coupled field theory and in the IIB string dual (the Polchinski-Strassler background [35]).

The present investigation is a natural extension of these lines of research, in particular along the second and third. Namely, we study the stability of the semi-local vortices in the presence of an $\mathcal{N} = 1$ perturbation which makes the vortices non-BPS. In the case of a theory with a mass gap, the 't Hooft standard classification of possible massive phases [36] is applicable; as a result, there is a very clear relation between the Higgs mechanism and magnetic confinement. In the model discussed in this paper, this is not a priori clear due to the presence of massless degrees of freedom (see Ref. [37] for some of the subtleties associated with the massless case).

The question is important from the point of view that the vortex system (with gauge group H) being studied is actually a low-energy approximation of an underlying system with a larger gauge group, G. The homotopy-map argument shows that the regular monopoles arising from the symmetry breaking $G \to H$ are actually unstable in the full theory, when the low-energy VEVs breaking completely the gauge group are taken into account. In other words, such monopoles are actually confined by the vortices developed in the low-energy H system: this allows us to relate the continuous moduli and group transformation properties of the vortices to those of the monopoles appearing at the ends, thus explaining the origin of the dual gauge groups, such as the Goddard-Nuyts-Olive-Weinberg (GNOW) duals [38]. The stability problem on non-BPS non-Abelian strings have also been studied in another system: the Seiberg-dual theory of the $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ QCD [39].

In these considerations a subtle but crucial point is that when small terms arising from the symmetry breaking $G \to H$ are taken into account, neither the high energy system (describing the symmetry breaking $G \to H$ and regular monopoles) nor the low-energy system (in the Higgs phase $H \to \mathbf{1}$ describing the vortices) is BPS-saturated any longer. This on the one hand allows the monopoles and vortices to be related in a one-to-one map (each vortex ends up on a monopole); on the other hand, the system is no longer BPS and we must carefully check the fate of the moduli of the BPS vortices which do not survive the perturbation. In fact, zero modes related to global symmetries (orientational modes) still survive, because of their origin as Goldstone bosons, while other modes are no more protected by supersymmetry. These issues are the subject for investigation in this paper.

2 Stability of Non-Abelian Semi-local Vortices

2.1 Semi-local Vortices

The question of stability of the semi-local vortices [21, 22] for Abelian systems has been investigated by Hindmarsh [40] and other authors [41, 42], some time ago. The model which has been

studied is an Abelian Higgs system with more than one flavor ($N_{\rm f} > 1$),

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu \phi (\mathcal{D}^\mu \phi)^\dagger - \frac{\lambda}{2} \left(\phi\phi^\dagger - \xi\right)^2 \ , \tag{2.1}$$

where $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ is the standard covariant derivative, $\phi = (\phi_1, \phi_2, \ldots, \phi_{N_{\rm f}})$ represents a set of complex scalar matter fields of the same charge. This model is sometimes called the *semi-local* model since not all global symmetries, i.e. $U(N_{\rm f})$ here, are gauged. As a consequence, the vacuum manifold $\mathcal{M}$ is $\mathbb{C}P^{N_{\rm f}-1} = SU(N_{\rm f})/[SU(N_{\rm f}-1) \times U(1)]$. Since the first homotopy group of $\mathcal{M}$ is trivial for $N_{\rm f} > 1$, vortex solutions are not necessarily stable. For $\beta \equiv \lambda/e^2 < 1$ (i.e. type I superconductors) the vortex of ANO-type [43, 44] is found to be stable. In the interesting special (BPS) case, $\beta = 1$, there is a family of vortex solutions with the same tension, $T = 2\pi\xi$. Except for the special values of the moduli (in the space of solutions), which represent the ANO vortex (sometimes called a "local vortex"), the vortex has a power-like tail in the profile function, hence the vortex width (thickness of the string) can be of an arbitrary size*. In the limit of large size, the vortex essentially reduces to the $\mathbb{C}P^{N_f-1}$ sigma-model lump (or two-dimensional skyrmion), characterized by $\pi_2(\mathbb{C}P^{N_f-1}) = \mathbb{Z}$.

For $\beta > 1$ (i.e. type II superconductors), vortices are found to be unstable against fluctuations of the extra fields (flavors) which increase the size (and spread out the flux).

Properties of non-BPS solutions in the case of non-Abelian vortices, within the models similar to Eq. (2.1) but with a $U(N_{\rm c})$ gauge group, have been studied by some of us [24]. The interactions among the vortices are found to depend on the relative orientations carried by these non-Abelian vortices as well as the distances between the vortices. The two new regimes† called type I*/II* have been found in addition to the usual type I/II superconductors [24].

The same authors investigated furthermore another class of (type I/I*) systems in a super Yang-Mills theory as well [23], characterized by certain non-vanishing adjoint scalar masses. These models are potentially important as they are exactly the sort of systems arising as the low-energy approximation as a result of symmetry breaking at some higher mass scale. Under such circumstances, the properties of the vortices in the low-energy system are closely related to those of the regular monopoles arising at high energies. With this motivation in mind, we shall here concentrate on this class of non-BPS vortices.

2.2 The Model and the Vortex Solution

The model we consider is an $\mathcal{N} = 2$ supersymmetric ($U(N_{\rm c})$) gauge theory with $N_{\rm f} > N_{\rm c}$ hypermultiplets (quarks) $Q_f, \tilde{Q}_f$ ($f = 1, \ldots, N_{\rm f}$). Since all the essential features are already present in the minimal case i.e. $N_{\rm c} = 2$, we will concentrate in the following on this case, i.e., the $U(2)$ model‡. The superpotential at hand is

$$W = \frac{1}{\sqrt{2}} \left[\tilde{Q}_f(a_0 + a_i\tau^i + \sqrt{2}m_f)Q_f + W_e(a_0) + W_g(a_i\tau^i)\right] \ , \tag{2.2}$$

*This type of vortex solutions has been termed "semi-local vortices". Though it is not an entirely adequate term, we shall stick to it, as it is commonly used in the literature.

†At large distance, the type I* interaction is attractive for parallel vortices and repulsive for anti-parallel vortices. Vice versa for type II* interaction.

‡ The extension to more general cases, $N_{\rm c} > 2$, will be discussed in appendix A.

where

$$W_e = -\xi\, a_0 + \eta\, a_0^2 \ , \quad W_g(a_i\tau^i) = \mu\, a_i a_i \ . \tag{2.3}$$

Two real positive mass parameters η and μ have been introduced for the adjoint scalars. The terms proportional to η, μ break $\mathcal{N}=2$ supersymmetry (SUSY) to $\mathcal{N}=1$. m_f are the (bare) quark masses. ξ is the F-term Fayet-Iliopoulos (FI) parameter. In fact, it is $SU(2)_R$-equivalent to the standard FI term. If only the ξ term is kept, the system remains $\mathcal{N}=2$ supersymmetric.

This kind of system naturally arises in the $\mathcal{N}=2$, $SU(3)$ SQCD, with SUSY softly broken down to $\mathcal{N}=1$ with a mass term of the form

$$W = \kappa\, \mathrm{Tr}\Phi^2 \ . \tag{2.4}$$

Indeed, when the bare masses for the squarks are tuned to some special values, there exist quantum vacua in which the non-Abelian gauge symmetry $SU(2)\times U(1)$ is preserved (the so-called $r=2$ vacua) [45]. The low energy effective theory in these vacua is exactly the theory we are studying here. As the quantum vacua with an $SU(2)$ magnetic gauge group require the presence of a sufficient number of flavors, $N_f \geq 2\,r = 4$, we must necessarily deal with a system with an excess number of flavors ($N_f > N_c$). As these systems in the BPS approximation ($\eta, \mu = 0$) contain semi-local vortices with arbitrarily large widths, the necessity of studying the fate of these vortices in the presence of the perturbations η, μ presents itself quite naturally.

The bosonic part of the Lagrangian is (we use the same symbols for the scalars as for the corresponding superfields):

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4g^2}(F^i_{\mu\nu})^2 - \frac{1}{4e^2}(F^0_{\mu\nu})^2 + \frac{1}{g^2}|\mathcal{D}_\mu a_i|^2 + \frac{1}{e^2}|\partial_\mu a_0|^2 \\ &+ (\mathcal{D}_\mu Q_f)^\dagger \mathcal{D}^\mu Q_f + \mathcal{D}_\mu \tilde{Q}_f (\mathcal{D}^\mu \tilde{Q}_f)^\dagger - V(Q,\tilde{Q},a_i,a_0) \ ,\end{aligned} \tag{2.5}$$

where e is the $U(1)$ gauge coupling and g is the $SU(2)$ gauge coupling. The covariant derivatives and field strengths, respectively, are defined by

$$\begin{aligned}&\mathcal{D}_\mu \left(Q_f, \tilde{Q}^*_f\right) = \left(\partial_\mu - i A^i_\mu \frac{\tau^i}{2} - \frac{i}{2}A^0_\mu\right)\left(Q_f, \tilde{Q}^*_f\right) \ , \quad \mathcal{D}_\mu a_i = \partial_\mu a_i + \epsilon^{ijk} A^j_\mu a_k \ , \\ &F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu \ , \quad F_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu \ .\end{aligned} \tag{2.6}$$

The potential V is the sum of the following D and F terms

$$\begin{aligned}V &= V_1 + V_2 + V_3 + V_4 \ , \\ V_1 &= \frac{g^2}{8}\left(\frac{2}{g^2}\epsilon^{ijk}\bar{a}_j a_k + \mathrm{Tr_f}[Q^\dagger \tau^i Q] - \mathrm{Tr_f}[\tilde{Q}\tau^i\tilde{Q}^\dagger]\right)^2 \ , \\ V_2 &= \frac{e^2}{8}\left(\mathrm{Tr_f}[Q^\dagger Q] - \mathrm{Tr_f}[\tilde{Q}\tilde{Q}^\dagger]\right)^2 \ , \\ V_3 &= \frac{g^2}{2}\left|\mathrm{Tr_f}[\tilde{Q}\tau^i Q] + 2\,\mu a_i\right|^2 + \frac{e^2}{2}\left|\mathrm{Tr_f}[\tilde{Q}Q] - \xi + 2\,\eta a_0\right|^2 \ , \\ V_4 &= \frac{1}{2}\sum_{f=1}^{N_\mathrm{f}}\left|(a_0 + \tau^i a_i + \sqrt{2}m_f)Q_f\right|^2 + \frac{1}{2}\sum_{f=1}^{N_\mathrm{f}}\left|(a_0 + \tau^i a_i + \sqrt{2}m_f)\tilde{Q}^\dagger_f\right|^2 \ ,\end{aligned} \tag{2.7}$$

where $\mathrm{Tr_f}$ denotes a trace over the flavor indices. The squark multiplets are kept massless in the remainder of the paper

$$m_f = 0 \ . \tag{2.8}$$

The theory has a degenerate set of vacua in the Higgs phase where the gauge symmetry is completely broken, which is the cotangent bundle over the complex Grassmanian manifold $\mathcal{M}_{\mathrm{vac}} = Gr_{N_\mathrm{f},N_\mathrm{c}} \simeq SU(N_\mathrm{f})/[SU(N_\mathrm{c}) \times SU(N_\mathrm{f}-N_\mathrm{c}) \times U(1)]$ [46] with $N_\mathrm{c} = 2$. It is important to notice that the first homotopy group of the Grassmanian manifold is trivial (for $N_\mathrm{f} > N_\mathrm{c}$). Up to gauge and flavor rotations, we can choose the following VEV for the scalar fields

$$Q = \tilde{Q}^\dagger = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \ , \quad a_0 = 0 \ , \quad a = 0 \ , \tag{2.9}$$

where $a \equiv a_i \tau^i/2$. The vacuum is invariant under the following global color-flavor locked rotations ($U_\mathrm{c} \in SU(2)$, $U_\mathrm{f} \in SU(2) \subset SU(N_\mathrm{f})$)[§]:

$$Q \to U_\mathrm{c} Q U_\mathrm{f}^\dagger \ , \quad \tilde{Q} \to U_\mathrm{f}^\dagger \tilde{Q} U_\mathrm{c} \ , \quad a \to U_\mathrm{c} a U_\mathrm{c}^\dagger \ , \quad F_{\mu\nu} \to U_\mathrm{c} F_{\mu\nu} U_\mathrm{c}^\dagger \ . \tag{2.10}$$

For $\eta \neq 0$, the theory has also another classical vacuum in the Coulomb phase

$$Q = \tilde{Q}^\dagger = 0 \ , \quad a_0 = \frac{\xi}{2\eta} \ , \quad a = 0 \ , \tag{2.11}$$

which "runs away" to infinity for $\eta = 0$. In what follows, we consider the vacuum (2.9) and the non-Abelian vortices therein.

Next, we will construct a particular solution for the fundamental (i.e. the minimum winding) vortex, simply by embedding the well-known solution for $N_\mathrm{f} = 2$ flavors in our model. In the next section the stability of this solution under perturbations of the additional flavors will be studied. Setting $\tilde{Q} = Q^\dagger$, the Euler-Lagrange equations for the theory are

$$\begin{aligned} \partial_\mu F_0^{\mu\nu} &= -ie^2 \left(Q_f^\dagger \mathcal{D}^\nu Q_f - (\mathcal{D}^\nu Q_f)^\dagger Q_f \right) \ , \\ \mathcal{D}_\mu F_i^{\mu\nu} &= -ig^2 \left(Q_f^\dagger \tau_i \mathcal{D}^\nu Q_f - (\mathcal{D}^\nu Q_f)^\dagger \tau_i Q_f \right) - \epsilon_{ijk} \left(a_j (\mathcal{D}^\nu a_k)^\dagger + \bar{a}_j \mathcal{D}^\nu a_k \right) \ , \\ \mathcal{D}^\mu \mathcal{D}_\mu Q &= -\frac{1}{2} \frac{\delta V}{\delta Q^\dagger} \ , \qquad \partial^\mu \partial_\mu a_0 = -e^2 \frac{\delta V}{\delta \bar{a}_0} \ , \qquad \mathcal{D}^\mu \mathcal{D}_\mu a_i = -g^2 \frac{\delta V}{\delta \bar{a}_i} \ . \end{aligned} \tag{2.12}$$

We start from the standard Ansatz for the local vortex embedded in the model with additional flavors

$$\begin{aligned} Q &= \begin{pmatrix} \phi_0(r) e^{i\theta} & 0 & 0 & \ldots & 0 \\ 0 & \phi_1(r) & 0 & \ldots & 0 \end{pmatrix} \ , \quad a_0 = \lambda_0(r) \ , \quad a = \lambda_1(r) \frac{\tau^3}{2} \ , \\ A_i &= A_i^\alpha \frac{\tau^\alpha}{2} = -\frac{\epsilon_{ij} x_j}{r^2} \left[1 - f_1(r) \right] \frac{\tau^3}{2} \ , \quad A_i^0 = -\frac{\epsilon_{ij} x_j}{r^2} \left[1 - f_0(r) \right] \ , \end{aligned} \tag{2.13}$$

with $r^2 = x_1^2 + x_2^2$. Notice that the adjoint fields $a_0, a = a_i \tau^i/2$ are non-trivial when we consider the non-BPS corrections, whereas they vanish everywhere if η, μ are set to zero (i.e. the BPS limit).

[§] The vacuum is also invariant under pure $SU(N_\mathrm{f} - 2)$ flavor rotations which act on the last $N_\mathrm{f} - 2$ columns.

By plugging the ansatz (2.13) into Eq. (2.12), we get a set of complicated second order differential equations for the six functions $\{\phi_0, \phi_1, f_0, f_1, \lambda_0, \lambda_1\}$ [23]. It is summarized in Appendix A, see Eqs. (A.14)-(A.19) with $N_c = 2$. This expression must be solved with the appropriate boundary conditions

$$\begin{aligned} f_1(0) = 1, \quad f_0(0) = 1, \quad f_1(\infty) = 0, \quad f_0(\infty) = 0, \\ \phi_0(\infty) = 1, \quad \phi_1(\infty) = 1, \quad \lambda_0(\infty) = 0, \quad \lambda_1(\infty) = 0. \end{aligned} \tag{2.14}$$

We find the following behavior for small r

$$\phi_0 \propto \mathcal{O}(r), \;\; \phi_1 \propto \mathcal{O}(1), \;\; \lambda_0 \propto \mathcal{O}(1), \;\; \lambda_1 \propto \mathcal{O}(1). \tag{2.15}$$

A numerical solution is shown in Fig. 1.

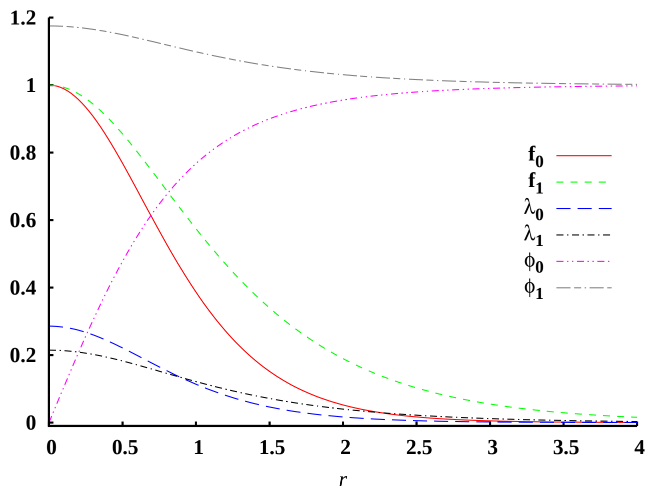


Figure 1: Profile functions for the vortex in the radial direction r: f_0 (unbroken/red), f_1 (short-dashed/green), λ_0 (long-dashed/blue), λ_1 (dash-dotted/black), ϕ_0 (dash-double-dotted/magenta), ϕ_1 (long-dash short-dash-dotted/grey). The numerical values are taken as $\xi = 2, e = 2, g = 1, \eta = 0.2, \mu = 0.4$. For this configuration $T = 0.964 T_{\rm BPS}$.

Further solutions can be obtained by acting with the $SU(2)_{\rm c+f}$ symmetry (2.10) on this Ansatz. As a consequence it develops some internal (orientational) zero modes. In fact, the vortex leaves a $U(1)_{\rm c+f}$ subgroup of $SU(2)_{\rm c+f}$ unbroken, i.e. these zero modes parameterize the space $\mathbb{C}P^1 = SU(2)/U(1) = S^2$.

In the BPS limit $\eta = \mu = 0$, the system has vortices which are solutions to the first-order equations [3], see also Eqs. (A.23) and (A.24), with tension

$$T_{\rm BPS} = 2\pi\xi \;. \tag{2.16}$$

As $\lambda_0(x) = \lambda_1(x) = 0$ ($a_0 = a = 0$) in this BPS configuration, its substitution into the Lagrangian (2.5) gives the BPS value, $T_{\rm BPS} = 2\pi\xi$, even if it is no longer a solution to the vortex equations (2.12), for $\eta, \mu \neq 0$. This means that the non-BPS vortex tension derived from Eq. (2.12) is necessarily *less* than the BPS value. In other words, the model we are considering always yields type I superconductivity [23].

Our result below – the stability of the ANO-like vortices – thus generalizes naturally the known correlation between the type of superconductor and the kinds of stable vortices, found in the Abelian Higgs models [40, 41, 42]. Also, in so much as we study a softly-broken $\mathcal{N}=2$ supersymmetric gauge theory and its low-energy vortex system, the present work can be regarded as an extension of the one in $SU(2)$ gauge theory [47], even though in the latter the low-energy system was naturally Abelian ($U(1)$). We have verified that the same qualitative conclusion holds in that case.

2.3 Fluctuation Analysis

In this section, we will generalize the fluctuation analysis of the stability of the Abelian vortex of Ref. [40] to the non-Abelian vortex. As we have mentioned, the first homotopy group of the vacuum manifold is trivial, thus the local vortex found in the previous section can be unstable. To study the stability of non-BPS local vortices, we must consider the quadratic variations of the Lagrangian due to small perturbations of the background fields. It turns out that the quadratic variation of the Lagrangian can be written as the sum of two pieces

$$\delta^2\mathcal{L} = \delta^2\mathcal{L}\big|_{\text{local fields}} + \delta^2\mathcal{L}\big|_{\text{semi}-\text{local fields}} \ , \tag{2.17}$$

where the first term denotes the variation with respect to the fields describing a non-trivial background vortex configuration (i.e. the "local" fields), while the second term is the variation with respect to the "semi-local" fields ($Q_f, \tilde{Q}_f$ with $N_\mathrm{f} \geq f > N_\mathrm{c} = 2$). The key point is that there are no mixed terms (at the second order) between the variations of the background fields and the "semi-local" fields. The first term cannot give rise to instabilities (i.e. the local vortices with $N_\mathrm{f} = 2$ are topologically stable). Hence, it suffices for our purpose to study only the second term

$$\delta^2\mathcal{L}\big|_{\text{semi}-\text{local fields}} = (\mathcal{D}_{\mathrm{b}\mu}\delta Q_3)^\dagger \mathcal{D}_\mathrm{b}^\mu \delta Q_3 + (\mathcal{D}_{\mathrm{b}\mu}\delta\tilde{Q}_3)(\mathcal{D}_\mathrm{b}^\mu\delta\tilde{Q}_3)^\dagger - \delta^2 V\big|_{Q_3,\tilde{Q}_3} \ , \tag{2.18}$$

where we have taken, for simplicity and without loss of generality, $N_\mathrm{f} = 3$. The subscript 'b' denotes background fields which we fix to be given by the Ansatz (2.13) in the following. Let us work out this variation explicitly. Even if ${\tilde{Q}_\mathrm{b}}^\dagger = Q_\mathrm{b}$ in the background fields we keep their variations independent.

Let us first consider the variation of the potential (2.7), piece by piece:

Variation of V_1

Remember that we are only interested in the variations that involve the semi-local fields, i.e. the variation with respect to the third flavor. The crucial observation is that the background value for this field is zero: $\tilde{Q}^\dagger_{3\mathrm{b}} = Q_{3\mathrm{b}} = 0$. This means that the variation of terms such as $\mathrm{Tr}_\mathrm{f}[Q^\dagger \tau^i Q]$ are already quadratic in the perturbation. Furthermore, the term $\epsilon^{ijk}\bar{a}_j a_k$ evaluated on the background fields is zero, due to a_3 being the only non-zero non-Abelian adjoint field. Thus, the variation of V_1 with respect to the semi-local fields is at least cubic, coming from the cross terms involving the adjoint field. Hence, the quadratic variation vanishes

$$\delta^2 V_1\big|_{Q_3,\tilde{Q}_3} = 0 \ . \tag{2.19}$$

Variation of V_2

V_2 contains no adjoint fields. The same argument used for V_1 goes through: the variations of V_2 are at least quartic

$$\delta^2 V_2\big|_{Q_3,\tilde{Q}_3} = 0 \ . \tag{2.20}$$

Variation of V_3

The variation of V_3 gives us the first non-trivial contribution. In this case the variation with respect to the semi-local fields is at least quadratic, and it is given by

$$\begin{aligned}\delta^2 V_3\big|_{Q_3,\tilde{Q}_3} = & \frac{g^2}{2}\left(\mathrm{Tr}_{\mathrm{f}}(Q_{\mathrm{b}}^\dagger \tau^3 Q_{\mathrm{b}}) + 2\,\mu a_{3\mathrm{b}}\right)\left(\delta\tilde{Q}_3\tau^3\delta Q_3 + \mathrm{c.c.}\right) \\ & + \frac{e^2}{2}\left(\mathrm{Tr}_{\mathrm{f}}(Q_{\mathrm{b}}^\dagger Q_{\mathrm{b}}) - \xi + 2\,\eta a_{0\mathrm{b}}\right)\left(\delta\tilde{Q}_3\delta Q_3 + \mathrm{c.c.}\right) \ .\end{aligned} \tag{2.21}$$

Variation of V_4

The variation of V_4 is also quadratic and is simply

$$\delta^2 V_4\big|_{Q_3,\tilde{Q}_3} = \frac{1}{2}\delta Q_3^\dagger\left(a_{0\mathrm{b}} + \tau^3 a_{3\mathrm{b}}\right)^2 \delta Q_3 + \frac{1}{2}\delta\tilde{Q}_3\left(a_{0\mathrm{b}} + \tau^3 a_{3\mathrm{b}}\right)^2 \delta\tilde{Q}_3^\dagger \ . \tag{2.22}$$

The variations are not diagonal, thus involve mixed terms like $\delta\tilde{Q}_3\delta Q_3$. We can easily diagonalize them by the following change of coordinates (keeping the kinetic terms canonical)

$$\delta Q_3 = \frac{1}{\sqrt{2}}(q + \tilde{q}^\dagger) \ , \qquad \delta\tilde{Q}_3 = \frac{1}{\sqrt{2}}(q^\dagger - \tilde{q}) \ , \tag{2.23}$$

which yields the following variation of the potential to second order

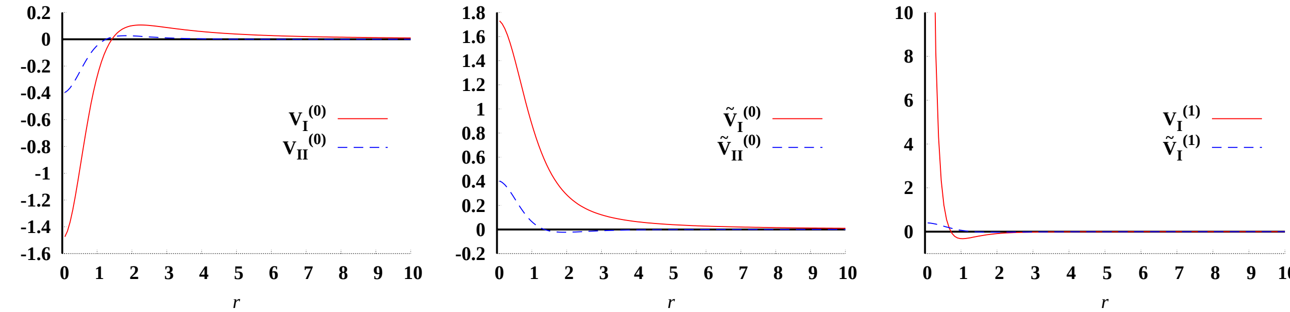
$$\begin{aligned}\delta^2 V\big|_{Q_3,\tilde{Q}_3} = & \frac{g^2}{2}\left(\mathrm{Tr}_{\mathrm{f}}(Q_{\mathrm{b}}^\dagger\tau^3 Q_{\mathrm{b}}) + 2\,\mu a_{3\mathrm{b}}\right)\left(|q_I|^2 - |\tilde{q}_I|^2 - |q_{II}|^2 + |\tilde{q}_{II}|^2\right) \\ & + \frac{e^2}{2}\left(\mathrm{Tr}_{\mathrm{f}}(\tilde{Q}_{\mathrm{b}} Q_{\mathrm{b}}) - \xi + 2\,\eta a_{0\mathrm{b}}\right)\left(|q_I|^2 - |\tilde{q}_I|^2 + |q_{II}|^2 - |\tilde{q}_{II}|^2\right) \\ & + \frac{1}{2}\left(a_{0\mathrm{b}} + a_{3\mathrm{b}}\right)^2\left(|q_I|^2 + |\tilde{q}_I|^2\right) + \frac{1}{2}\left(a_{0\mathrm{b}} - a_{3\mathrm{b}}\right)^2\left(|q_{II}|^2 + |\tilde{q}_{II}|^2\right) \ ,\end{aligned} \tag{2.24}$$

where the capital indices I and II label the color components. The problem of stability is now reduced to studying four decoupled Schrödinger equations. We expand the fluctuations as

$$q_{I,II} \equiv \sum_k \psi_{I,II}^{(k)} e^{ik\theta} \ , \quad \tilde{q}_{I,II} \equiv \sum_k \tilde{\psi}_{I,II}^{(k)} e^{-ik\theta} \ . \tag{2.25}$$

Using these expansions, the quadratic variations of the energy density (2.18) give rise to the following Schrödinger equations

$$\begin{aligned}\left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + V_{I,II}^{(k)}\right]\psi_{I,II}^{(k)} &= M_{I,II}^{(k)}\psi_{I,II}^{(k)} \ , \\ \left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \tilde{V}_{I,II}^{(k)}\right]\tilde{\psi}_{I,II}^{(k)} &= \tilde{M}_{I,II}^{(k)}\tilde{\psi}_{I,II}^{(k)} \ ,\end{aligned} \tag{2.26}$$



(a) (b) (c)

Figure 2: Potentials as functions of r: (a) $V_I^{(0)}$ and $V_{II}^{(0)}$. (b) $\tilde{V}_I^{(0)}$ and $\tilde{V}_{II}^{(0)}$. (c) $V_I^{(1)}$ and $\tilde{V}_I^{(1)}$. The results are plotted for $e = 2, g = 1, \xi = 2, \eta = 0.2$ and $\mu = 0.4$.

which must be solved with the following boundary conditions at small r:

$$\psi_{I,II}^{(k)}(r), \tilde{\psi}_{I,II}^{(k)}(r) = r^k + \mathcal{O}(r^{k+1}). \tag{2.27}$$

The effective potentials are

$$V_{I,II}^{(k)} = \frac{1}{4r^2}\left[f_0 - 1 \pm (f_1 - 1) + 2k\right]^2 + \frac{1}{2}(\lambda_0 \pm \lambda_1)^2 \tag{2.28}$$

$$+ \frac{e^2}{2}(\phi_0^2 + \phi_1^2 - \xi + 2\,\eta\lambda_0) \pm \frac{g^2}{2}(\phi_0^2 - \phi_1^2 + 2\,\mu\lambda_1)$$

$$\tilde{V}_{I,II}^{(k)} = \frac{1}{4r^2}\left[f_0 - 1 \pm (f_1 - 1) + 2k\right]^2 + \frac{1}{2}(\lambda_0 \pm \lambda_1)^2 \tag{2.29}$$

$$- \frac{e^2}{2}(\phi_0^2 + \phi_1^2 - \xi + 2\,\eta\lambda_0) \mp \frac{g^2}{2}(\phi_0^2 - \phi_1^2 + 2\,\mu\lambda_1)\ . \tag{2.30}$$

The upper signs refer to the color-index I, while the lower to index II. Since only the first terms are dependent on k, it is easy to check that

$$\left(V_I^{(k)}, V_{II}^{(k)}\right) \geq \left(\min\left[V_I^{(0)}, V_I^{(1)}\right], V_{II}^{(0)}\right)\ , \quad \left(\tilde{V}_I^{(k)}, \tilde{V}_{II}^{(k)}\right) \geq \left(\min\left[\tilde{V}_I^{(0)}, \tilde{V}_I^{(1)}\right], \tilde{V}_{II}^{(0)}\right)\ , \tag{2.31}$$

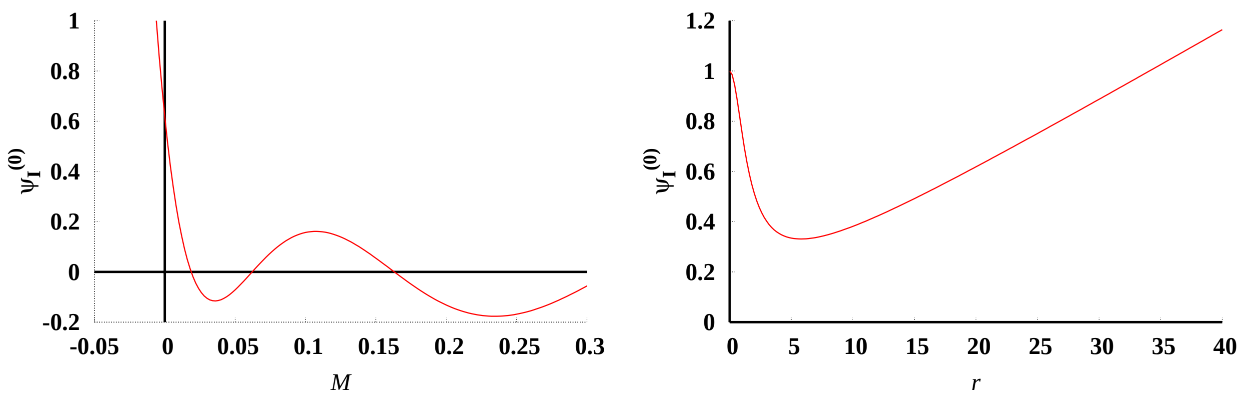
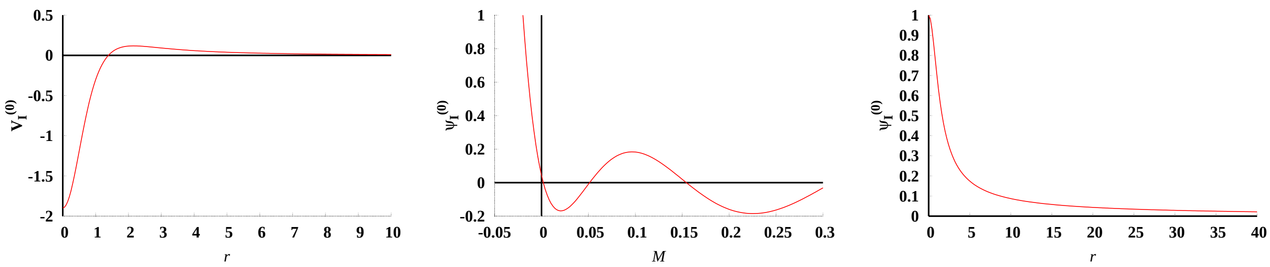


Figure 3: Left panel: the *value* of the wave function $\psi_I^{(0)}$ at large distance ($r = 20$, in the unit of the vortex size), as a function of M (see Eqs. (2.26)). There are no zeros at negative values of M: this implies the absence of normalizable tachyonic modes. Right panel: the wave function for the zero energy fluctuation: $M = 0$. The linear behavior is a common feature for zero energy wave functions. The numerical values of the parameters are chosen as in Fig. 2.



(a) (b) (c)

Figure 4: Numerical results in the BPS case: $e = 2$, $g = 1$, $\xi = 2$, $\eta = \mu = 0$. (a) The potential $V_I^{(0)}$ has a zero energy fluctuation. (b) The value of $\psi_I^{(0)}$ at a large distance ($r = 20$) as function of M. (c) The zero energy wave function $\psi_I^{(0)}$ as function of r.

for all k. To prove the above inequalities, it is sufficient to recall that the profile functions for the gauge fields satisfy: $0 \leq f_{0,1} \leq 1$.

From a well-known theorem in one-dimensional quantum mechanics, it follows that (valid for the lowest eigenvalues)

$$\left(M_I^{(k)}, M_{II}^{(k)}\right) \geq \left(\min\left[M_I^{(0)}, M_I^{(1)}\right], M_{II}^{(0)}\right) \quad , \quad \left(\tilde{M}_I^{(k)}, \tilde{M}_{II}^{(k)}\right) \geq \left(\min\left[\tilde{M}_I^{(0)}, \tilde{M}_I^{(1)}\right], \tilde{M}_{II}^{(0)}\right) \ . \tag{2.32}$$

Hence, in order to exclude the existence of negative eigenvalues, it suffices to study the six potentials: $V_I^{(0,1)}$, $\tilde{V}_I^{(0,1)}$, $V_{II}^{(0)}$ and $\tilde{V}_{II}^{(0)}$.

Fig. 2 shows that only the potentials $V_I^{(0)}$ and $V_{II}^{(0)}$ can be sufficiently negative, such that one could expect negative eigenvalues. The potentials $V_I^{(1)}$ and $\tilde{V}_I^{(1)}$ are even divergent at the core of the vortex. Indeed, the results of the numerical analysis show (up to the numerical precision) that there are no negative eigenvalues for any of the potentials, $V_I^{0,1}$, $\tilde{V}_I^{0,1}$, V_{II}^0 and $\tilde{V}_{II}^0$. We have checked this statement in a very wide range of values of the couplings of the theory. This strongly suggests the absence of negative eigenvalues for all values of the couplings. The result is shown in Fig. 3 for a particular choice of parameters. Note that all the potentials go to zero at large distance. This means that there will be a set of positive eigenvalues that represent the continuum states, which can be interpreted as interaction modes with the massless Goldstone bosons of the bulk theory.

The status of the zero energy wave functions however is subtler. The wave functions for the non-BPS (Fig. 3) and the BPS cases (Fig. 4) are both non-normalizable, hence they should both be interpreted as a part of the continuum. However, there is an important difference between the two cases. In the BPS case, the wave function is limited and goes to zero at infinity; its squared norm diverges only logarithmically with the transverse volume ($\sim \log L^2$). In the non-BPS case, the wave function is unlimited, and it diverges linearly (Fig. 3). This implies a divergence of its squared norm being quadratic in the transverse volume ($\sim L^4$). Such a fluctuation changes the vacuum expectation values of the scalars at infinity, thus does not correspond to a size zero mode (collective coordinate). The situation is clearer when our vortex system is put into a finite volume. In the BPS case, the wave function of the semi-local mode asymptotically approaches zero and is an (approximately) acceptable wave function. In a box, it represents a zero energy bound state. On the contrary, the wave function for the non-BPS case is non-normalizable. The corresponding zero energy state is eliminated and the semi-local excitations have a mass gap.

This observation strongly suggests that in the non-BPS case the size moduli disappear.

As a further check, we note that the potential which gives rise to the size zero modes in the BPS limit (which we know exist in that case) is precisely $V_I^{(0)}$. The zero mode is thus given by a fluctuation of the field $\psi_I^{(0)}$. Using Eq. (2.23) we see that $\delta Q_{3I} = \delta \tilde{Q}_{3I} = \sqrt{2}\psi_I^{(0)}$, while all other components of Q_3 and $\tilde{Q}_3$ vanish. This is exactly the fluctuation we expect for the semi-local vortex, which in the BPS limit is known to have the following form [19, 20]:

$$Q = \begin{pmatrix} \phi_0(r)e^{i\theta} & 0 & \chi(r) \\ 0 & \phi_1(r) & 0 \end{pmatrix} . \tag{2.33}$$

3 Meta-stability versus Absolute Stability

The results of the previous section indicate that the local (ANO-like) vortex embedded in a model with additional flavors is a local minimum of the action. The problem whether or not this solution is a global minimum cannot be addressed with a fluctuation analysis only. In principle, the existence of an instability related to large fluctuations of the fields is still not excluded.

However, we have a reason to believe that the local vortex is truly stable, being the global minimum of the action. Note that a vortex cannot dilute in the whole space. In fact, such diluted configuration would have an energy that is equal to the BPS bound, $T_\infty = 2\pi\xi = T_{\rm BPS}$, while the tension of the local vortices, in the class of theories we are considering, is always less: $T < T_{\rm BPS}$, as already noted in Section 2.2. To see this, we will first show that the semi-local vortex solution, obtained in the BPS limit, *is an approximate solution to the non-BPS equations*, in the limit of an infinite size of the vortex. Then, it is easy to check that the tension of such approximate configuration converges to the BPS value. Let us write the equations of motion for the semi-local configuration (2.33)

$$\begin{aligned}
f_0'' - \frac{f_0'}{r} &= e^2\left(f_1(\phi_0^2 - \phi_1^2 + \chi^2) + f_0(\phi_0^2 + \phi_1^2 + \chi^2) - 2\chi^2\right) , \\
f_1'' - \frac{f_1'}{r} &= g^2\left(f_1(\phi_0^2 + \phi_1^2 + \chi^2) + f_0(\phi_0^2 - \phi_1^2 + \chi^2) - 2\chi^2\right) , \\
\phi_0'' + \frac{\phi_0'}{r} - \frac{(f_0+f_1)^2\phi_0}{4r^2} &= \frac{\phi_0}{2}\left((\lambda_0+\lambda_1)^2 + e^2 A + g^2 B\right) , \\
\phi_1'' + \frac{\phi_1'}{r} - \frac{(f_0-f_1)^2\phi_1}{4r^2} &= \frac{\phi_1}{2}\left((\lambda_0-\lambda_1)^2 + e^2 A - g^2 B\right) , \\
\chi'' + \frac{\chi'}{r} - \frac{(f_0+f_1-2)^2\chi}{4r^2} &= \frac{\chi}{2}\left((\lambda_0+\lambda_1)^2 + e^2 A + g^2 B\right) , \\
\lambda_0'' + \frac{\lambda_0'}{r} &= e^2\left((\lambda_0+\lambda_1)(\phi_0^2+\chi^2) + (\lambda_0-\lambda_1)\phi_1^2 + e^2\eta\, A\right) , \\
\lambda_1'' + \frac{\lambda_1'}{r} &= g^2\left((\lambda_0+\lambda_1)(\phi_0^2+\chi^2) - (\lambda_0-\lambda_1)\phi_1^2 + g^2\mu\, B\right) ,
\end{aligned} \tag{3.1}$$

where we have defined

$$A \equiv \phi_0^2 + \phi_1^2 + \chi^2 - \xi + 2\eta\lambda_0 , \quad B \equiv \phi_0^2 - \phi_1^2 + \chi^2 + 2\mu\lambda_1 . \tag{3.2}$$

For simplicity, we take the gauge couplings to be equal: $e = g$. Let us consider the following set of approximate solutions corresponding to a semi-local BPS vortex in the limit of large size

(lump limit): $a \gg 1/e\sqrt{\xi}$:

$$(\phi_0, \phi_1, \chi) = \sqrt{\frac{\xi}{2}} \left(\frac{r}{\sqrt{r^2 + |a|^2}}, 1, \frac{a}{\sqrt{r^2 + |a|^2}} \right) , \quad f_0 = f_1 = \frac{|a|^2}{r^2 + |a|^2} , \quad \lambda_0 = \lambda_1 = 0 . \quad (3.3)$$

The right-hand sides of Eqs. (3.1) calculated with this configuration vanish obviously. The left-hand sides are derivative terms which disappear in the large size limit: the set of fields above, trivially satisfy the fourth and the last two equations, while they satisfy the other equations up to terms of order $\mathcal{O}(1/|a|^2)$

$$\begin{aligned}
\left| f_{0,1}'' - \frac{f_{0,1}'}{r} \right| &= \frac{8|a|^2 r^2}{(r^2 + |a|^2)^3} \leq \frac{32}{27|a|^2} , \\
\left| \phi_0'' + \frac{\phi_0'}{r} - \frac{\phi_0 (f_0 + f_1)^2}{4r^2} \right| &= \sqrt{\frac{\xi}{2}} \frac{2|a|^2 r}{(|a|^2 + r^2)^{5/2}} \leq \sqrt{\frac{\xi}{2}} \left(\frac{4}{5} \right)^{\frac{5}{2}} \frac{1}{|a|^2} , \\
\left| \chi'' + \frac{\chi'}{r} - \frac{\chi(-2 + f_0 + f_1)^2}{4r^2} \right| &= \sqrt{\frac{\xi}{2}} \frac{2|a|^3}{(|a|^2 + r^2)^{5/2}} \leq \sqrt{\frac{\xi}{2}} \frac{2}{|a|^2} .
\end{aligned}$$

Thus, we see that the configuration of Eq. (3.3), in the limit $|a| \to \infty$, is a solution to the non-BPS equations of motion: it represents an infinitely diluted vortex, having the tension equal to the BPS value.

We thus see that an infinitely wide vortex is not favored energetically. The only remaining possibility is the existence of a true minimum for a vortex with some intermediate size, which involves also a non-trivial configuration for the semi-local fields. This case is very unlikely. In fact, we could not find any such solution in our numerical surveys. The relaxation method should have detected such an unexpected stable vortex.

We consider our results as a strong evidence for the fact that the local vortex is a true minimum, absolutely stable against becoming a semi-local vortex.

4 Low-energy Effective Theory

In this short section we only make some considerations on the low-energy effective theory of the vortex which follow from the results of the previous sections.

In the semi-local case, the vacuum has no mass gap, thus there are massless Nambu-Goldstone bosons in the bulk. This fact gives rise to subtleties (which we encountered in Sec. 2.3) when defining an effective theory for the vortex only. This is because both the vortex excitations and the massless particles in the bulk appear as light modes. As previously mentioned, semi-local excitations are non-normalizable and the corresponding states must be considered as bulk excitations. In a rigorous effective theory, we should take into account only localizable and normalizable modes, which discards the bulk excitations[¶]. This holds both in the BPS and non-BPS cases. In fact, effective actions which include semi-local excitations have already been proposed in the literature. The effective actions for BPS semi-local vortices derived in Ref. [19, 11], however, are valid only in a finite volume.

[¶]Similar issues have been already discussed for monopoles and vortices, see for example Refs. [14, 48, 49].

It seems to us that the following point has not been stressed clearly enough in the literature. We consider all the excitations related to non-normalizable zero energy wave functions as part of the massless states of the bulk theory. In the BPS case, the surviving supersymmetry enables us to describe these fluctuations as collective coordinates of the vortex. It is known, for example, that a single non-Abelian BPS vortex has non-normalizable size parameters (in the case $N_f > N_c$). From the point of view that we propose here, however, the effective action on the local vortex is given by the orientational part only (normalizable states). The other collective coordinates (size) give rise to zero energy fluctuations which represent bulk excitations.

While these considerations might just be a matter of interpretation in the BPS case, they could be important if one wants to push further the study of the non-BPS vortices we started in this work.

5 Conclusion

In this paper we have examined the stability of the non-Abelian vortices in the context of an $\mathcal{N} = 2$, $H = U(N_\mathrm{c})$ model with $N_\mathrm{f} > N_\mathrm{c}$ flavors and an $\mathcal{N} = 1$ perturbation. The particular perturbation chosen (the adjoint scalar mass terms) makes our vortices non-BPS. Local (ANO-like) vortices are found to be stable; the vortex moduli corresponding to the semi-local vortices (present in the BPS case) disappear, leaving intact the orientational moduli – $\mathbb{C}P^{N_\mathrm{c}-1}$ in this particular model – related to the exact symmetry of the system.

This conclusion seems to be very reasonable, as the narrow, ANO-like vortices are needed to eliminate the regular monopoles from the spectrum of the full system, in the case our H gauge theory arises as a low-energy approximation of an underlying G theory, after the symmetry breaking $G \to H$. If the semi-local vortex would have survived the non-BPS perturbations, the magnetic monopoles would be deconfined [19]. Happily, this is not the case; the Higgs phase of the (electric) low-energy theory is actually a magnetic confinement phase.

Acknowledgments

The authors thank David Tong for fruitful discussions. S. B. G. and W. V. are grateful to Department of Physics, Keio University and Department of Physics, Tokyo Institute of Technology for warm hospitality where part of this work has been done. The work of M.E. is supported by the Research Fellowships of the Japan Society for the Promotion of Science for Research Abroad.

A Extension to $U(N_\mathrm{c})$

In this appendix, we will extend the analysis of the vortices in $SU(2) \times U(1)/\mathbb{Z}_2$ theory to the $SU(N_\mathrm{c}) \times U(1)/\mathbb{Z}_{N_\mathrm{c}}$ with the $N_\mathrm{f} \geq N_\mathrm{c}$ squarks in the fundamental representation.

The $\mathcal{N} = 2$ multiplets are the following. The $SU(N_\mathrm{c})$ vector multiplet $(A_\mu, a = a_1 + ia_2)$, the $U(1)$ vector multiplet $(A_{0\mu}, a_0 = a_{0,1} + ia_{0,2})$ and the hypermultiplets $(Q, \tilde{Q})$ in the fundamental representation of $SU(N_\mathrm{c})$. The squark fields Q and $\tilde{Q}^\dagger$ are $N_\mathrm{c} \times N_\mathrm{f}$ matrices and their $U(1)$

charges are $+1/2$ and $-1/2$, respectively. The covariant derivatives are defined as

$$\mathcal{D}_\mu(Q,\tilde{Q}^\dagger) = (\partial_\mu - iA_\mu - iA_{0\mu}/2)(Q,\tilde{Q}^\dagger)\ , \quad \mathcal{D}_\mu a = \partial_\mu a - i[A_\mu, a]\ . \tag{A.1}$$

We partially break SUSY by adding mass terms for the adjoint scalars a and a_0 in the following. The bosonic part of the softly broken $\mathcal{N}=2$ Lagrangian $\mathcal{L}=K-V$ takes the form

$$\begin{aligned} K &= \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{2}{g^2}\mathcal{D}_\mu a_i \mathcal{D}^\mu a_i + \mathcal{D}_\mu Q(\mathcal{D}^\mu Q)^\dagger + \mathcal{D}_\mu \tilde{Q}^\dagger(\mathcal{D}^\mu \tilde{Q}^\dagger)^\dagger\right] \\ &- \frac{1}{4e^2}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{e^2}\partial_\mu a_{0,i}\partial^\mu a_{0,i}\ , \end{aligned} \tag{A.2}$$

$$\begin{aligned} V &= \mathrm{Tr}\left[-\frac{4}{g^2}[a_1,a_2]^2 + \frac{g^2}{4}\left\langle QQ^\dagger - \tilde{Q}^\dagger\tilde{Q}\right\rangle^2\right] + \frac{e^2}{8}\left(\mathrm{Tr}\,[QQ^\dagger - \tilde{Q}^\dagger\tilde{Q}]\right)^2 \\ &+ g^2\mathrm{Tr}\left[\left\langle Q\tilde{Q} + 2\mu(a_1+ia_2)\right\rangle\left\langle Q\tilde{Q}+2\mu(a_1+ia_2)\right\rangle^\dagger\right] \\ &+ \frac{e^2}{2}\left|\mathrm{Tr}\,[Q\tilde{Q}] - \frac{N_\mathrm{c}\xi}{2} + 2\eta(a_{0,1}+ia_{0,2})\right|^2 + 2\mathrm{Tr}\left[\left(QQ^\dagger + \tilde{Q}^\dagger\tilde{Q}\right)\left(a_i + \frac{1}{2}a_{0,i}\mathbf{1}_N\right)^2\right]\ . \end{aligned} \tag{A.3}$$

Our normalization is $\mathrm{Tr}\,[T^aT^b]=\delta^{ab}/2$ for the generators of $SU(N_\mathrm{c})$. The space-time metric is taken as $\eta_{\mu\nu}=(+,-,-,-)$. The masses of the adjoint scalars (a,a_0) are μ and η. e is the $U(1)$ gauge coupling and g is the $SU(N_\mathrm{c})$ gauge coupling. $\langle X\rangle$ for an $N\times N$ matrix X stands for the traceless part of X [‖]: $\langle X\rangle \equiv X - \frac{\mathrm{Tr}\,[X]}{N}\mathbf{1}_N$ and $\mathrm{Tr}\,[\langle X\rangle]=0$. We will choose the following vacuum in which to construct vortex solutions

$$Q=\tilde{Q}^\dagger = \sqrt{\frac{\xi}{2}}\,(\mathbf{1}_{N_\mathrm{c}}\ ,\mathbf{0})\ , \quad a=a_0=0\ . \tag{A.4}$$

A.1 The Vortex Equations

Let us make an Ansatz for the minimal winding vortex solution. First of all, let us take an $SU(N_\mathrm{c})$ generator

$$\mathbf{E} \equiv \frac{N_\mathrm{c}-1}{N_\mathrm{c}}\ \mathrm{diag}\left(1, -\frac{1}{N_\mathrm{c}-1}, \cdots, -\frac{1}{N_\mathrm{c}-1}\right) \in \mathfrak{su}(N_\mathrm{c}), \quad \mathrm{Tr}\,[\mathbf{E}^2] = \frac{N_\mathrm{c}-1}{N_\mathrm{c}}\ . \tag{A.5}$$

We make the following diagonal Ansatz

$$Q=\tilde{Q}^\dagger = \left(\mathrm{diag}\left(e^{i\theta}\phi_0(r),\phi_1(r),\cdots,\phi_1(r)\right),\mathbf{0}\right)\ , \tag{A.6}$$

$$A_{0i} = -\epsilon_{ij}\frac{2}{N}\frac{x_j}{r^2}(1-f_0(r))\ , \quad a_0 = a_{0,1} = \frac{2}{N}\lambda_0(r)\ , \tag{A.7}$$

$$A_i = -\epsilon_{ij}\frac{x_j}{r^2}(1-f_1(r))\mathbf{E}\ , \quad a=a_1=\lambda_1(r)\mathbf{E}\ . \tag{A.8}$$

Note $\left\langle Q\tilde{Q}+2\mu a\right\rangle = \left\langle QQ^\dagger + 2\mu a_1\right\rangle = (\phi_0^2-\phi_1^2+2\mu\lambda_1)\mathbf{E}$ holds.

‖ Useful identities: $\mathrm{Tr}\,[\langle X\rangle\langle Y\rangle] = \mathrm{Tr}\,[XY] - \frac{\mathrm{Tr}\,[X]\mathrm{Tr}\,[Y]}{N}\mathbf{1}_N$, $\langle X+Y\rangle = \langle X\rangle + \langle Y\rangle$.

Inserting this Ansatz into the Lagrangian above leaves us with an effective Lagrangian for the fields $f_0, f_1, \lambda_0, \lambda_1, \phi_0$ and ϕ_1:

$$\begin{aligned}
\tilde{\mathcal{L}} &= 2\pi r(\tilde{K} - \tilde{V}) \ , && \text{(A.9)}\\
\tilde{K} &= -\frac{N_c - 1}{N_c g^2}\frac{f_1'^2}{r^2} - \frac{2(N_c-1)}{N_c g^2}\lambda_1'^{\,2} - \frac{2}{N_c^2 e^2}\frac{f_0'^2}{r^2} - \frac{4}{N_c^2 e^2}\lambda_0'^{\,2} - 2\phi_0'^{\,2} - 2(N_c-1)\phi_1'^{\,2} \ , && \text{(A.10)}\\
\tilde{V} &= \frac{(N_c-1)g^2}{N_c}B^2 + \frac{e^2}{2}A^2 + \frac{4}{N_c^2}\phi_0^2\left(\lambda_0 + (N_c-1)\lambda_1\right)^2 + \frac{4(N_c-1)}{N_c^2}\phi_1^2\left(\lambda_0 - \lambda_1\right)^2 \\
&\quad + 2\frac{(f_0 + (N_c-1)f_1)^2\phi_0^2}{N_c^2 r^2} + 2(N_c-1)\frac{(f_0 - f_1)^2\phi_1^2}{N_c^2 r^2} \ . && \text{(A.11)}
\end{aligned}$$

where we have defined

$$\begin{aligned}
A &= \phi_0^2 + (N_c-1)\phi_1^2 - \frac{N_c}{2}\xi + \frac{4}{N_c}\eta\lambda_0 \ , && \text{(A.12)}\\
B &= \phi_0^2 - \phi_1^2 + 2\mu\lambda_1 \ . && \text{(A.13)}
\end{aligned}$$

The vortex equations are simply the Euler-Lagrange equations for $\tilde{\mathcal{L}}$:

$$\begin{aligned}
& f_0'' - \frac{f_0'}{r} - e^2\left[(f_0 + (N_c-1)f_1)\phi_0^2 + (N_c-1)(f_0 - f_1)\phi_1^2\right] = 0 \ , && \text{(A.14)}\\
& f_1'' - \frac{f_1'}{r} - \frac{2g^2}{N_c}\left[(f_0 + (N_c-1)f_1)\phi_0^2 - (f_0 - f_1)^2\phi_1^2\right] = 0 \ , && \text{(A.15)}\\
& \lambda_0'' + \frac{\lambda_0'}{r} - e^2\left[\frac{N_c\eta e^2}{2}A + \left(\lambda_0 + (N_c-1)\lambda_1\right)\phi_0^2 + (N_c-1)\left(\lambda_0 - \lambda_1\right)\phi_1^2\right] = 0 \ , && \text{(A.16)}\\
& \lambda_1'' + \frac{\lambda_1'}{r} - g^2\left[\mu g^2 B + \frac{2}{N_c}\left(\lambda_0 + (N_c-1)\lambda_1\right)\phi_0^2 - \frac{2}{N_c}\left(\lambda_0 - \lambda_1\right)\phi_1^2\right] = 0 \ , && \text{(A.17)}\\
& \phi_0'' + \frac{\phi_0'}{r} - \left[\frac{(N_c-1)g^2}{N_c}B + \frac{e^2}{2}A + \frac{2}{N_c^2}\left(\lambda_0 + (N_c-1)\lambda_1\right)^2 + \frac{(f_0 + (N_c-1)f_1)^2}{N_c^2 r^2}\right]\phi_0 = 0 \ , && \text{(A.18)}\\
& \phi_1'' + \frac{\phi_1'}{r} - \left[-\frac{g^2}{N_c}B + \frac{e^2}{2}A + \frac{2}{N_c^2}\left(\lambda_0 - \lambda_1\right)^2 + \frac{(f_0 - f_1)^2}{N_c^2 r^2}\right]\phi_1 = 0 \ . && \text{(A.19)}
\end{aligned}$$

These equations should be solved with the boundary conditions

$$\begin{aligned}
(f_0, f_1, \lambda_0, \lambda_1, \phi_0, \phi_1) &\to \left(0,0,0,0,\sqrt{\frac{\xi}{2}},\sqrt{\frac{\xi}{2}}\right) \ , \qquad \text{as} \quad r \to \infty \ , && \text{(A.20)}\\
(f_0, f_1, \lambda_0, \lambda_1, \phi_0, \phi_1) &\to \left(1,1,\mathcal{O}(1),\mathcal{O}(1),0,\mathcal{O}(1)\right) \ , \qquad \text{as} \quad r \to 0 \ . && \text{(A.21)}
\end{aligned}$$

All other solutions can be generated from this Ansatz by a flavor rotation.

When we turn off the parameters μ and η, the model recovers full $\mathcal{N} = 2$ SUSY and the vortices therein become BPS states. One of the common properties for various BPS states is that the EoMs can be reduced to first order differential equations. The energy density can be

rewritten as follows

$$
\begin{aligned}
\frac{\tilde{\mathcal{E}}_{\rm BPS}}{2\pi r} &= \frac{N_{\rm c}-1}{N_{\rm c}g^2}\left[\frac{f_1'}{r}-g^2\left(\phi_0^2-\phi_1^2\right)\right]^2+\frac{2}{N_{\rm c}^2e^2}\left[\frac{f_0'}{r}-\frac{N_{\rm c}e^2}{2}\left(\phi_0^2+(N_{\rm c}-1)\phi_1^2-\frac{N_{\rm c}}{2}\xi\right)\right]^2 \\
&\quad +2\left[\phi_0'-\frac{f_0+(N_{\rm c}-1)f_1}{N_{\rm c}r}\phi_0\right]^2+2(N_{\rm c}-1)\left[\phi_1'-\frac{f_0-f_1}{N_{\rm c}r}\phi_1\right]^2 \\
&\quad -\xi\frac{f_0'}{r}+\text{surface terms}\ , \\
&\geq -\xi\frac{f_0'}{r}\ , \qquad\qquad (\text{A.22})
\end{aligned}
$$

where we have discarded λ_0 and λ_1 because of $\mu=\eta=0$. The bound from below is saturated for the solutions satisfying the BPS equations

$$
\frac{f_1'}{r}=g^2\left(\phi_0^2-\phi_1^2\right)\ ,\quad \frac{f_0'}{r}=\frac{N_{\rm c}e^2}{2}\left(\phi_0^2+(N_{\rm c}-1)\phi_1^2-\frac{N_{\rm c}}{2}\xi\right)\ , \qquad (\text{A.23})
$$

$$
\phi_0'=\frac{f_0+(N_{\rm c}-1)f_1}{N_{\rm c}r}\phi_0\ ,\quad \phi_1'=\frac{f_0-f_1}{N_{\rm c}r}\phi_1\ . \qquad (\text{A.24})
$$

The tension of the BPS vortex is

$$
T=\int_0^\infty dr\ \tilde{\mathcal{E}}_{\rm BPS}=2\pi\big[-f_0'(r)\big]_0^\infty=2\pi\xi\ . \qquad (\text{A.25})
$$

A.2 The Perturbations

In order to study whether the vortex solution in the previous subsection is stable, we perturb the fields around the solution considered to be the background. Let us denote the squark fields by

$$
Q=(Q_{\rm b}+\delta Q_1,\delta Q_2)\ ,\quad \tilde{Q}^\dagger=(Q_{\rm b}+\delta\tilde{Q}_1^\dagger,\delta\tilde{Q}_2^\dagger)\ . \qquad (\text{A.26})
$$

Note $\delta Q_1,\delta\tilde{Q}_1$ are $N_{\rm c}\times N_{\rm c}$ and $\delta Q_2,\delta\tilde{Q}_2$ are $N_{\rm c}\times(N_{\rm f}-N_{\rm c})$. Since the last two are perturbations around a vanishing background, they are completely decoupled from all the other fields in the quadratic Lagrangian as we have seen in the main body of this paper. To unravel the mixed terms, we make the following redefinition of fields $\delta Q_2=(q+\tilde{q}^\dagger)/\sqrt{2}$ and $\delta\tilde{Q}_2=(q^\dagger-\tilde{q})/\sqrt{2}$ as before. In order to derive the usual Schrödinger-type equations, we expand $q,\tilde{q}$ as follows

$$
q=\sum_k e^{ik\theta}\begin{pmatrix}\psi_0^{(k)}(r)\\ \psi_1^{(k)}(r)\\ \vdots\\ \psi_{N_{\rm c}-1}^{(k)}(r)\end{pmatrix}\ ,\quad \tilde{q}=\sum_k e^{-ik\theta}\left(\tilde{\psi}_0^{(k)}(r),\ \tilde{\psi}_1^{(k)}(r),\cdots,\tilde{\psi}_{N_{\rm c}-1}^{(k)}(r)\right)\ , \qquad (\text{A.27})
$$

where $\psi_i^{(k)}$ is an $N_{\rm f}-N_{\rm c}$ row vector and $\tilde{\psi}_i^{(k)}$ is an $N_{\rm f}-N_{\rm c}$ column vector.

Substituting these into the Lagrangian (A.3), we obtain the following four effective Lagrangians

$$
\frac{\mathcal{L}^{(k;0)}}{2\pi r}=-(\partial_r\psi_{0,A}^{(k)})^2-V_0^{(k)}(r)(\psi_{0,A}^{(k)})^2\ ,\qquad \frac{\mathcal{L}^{(k;i)}}{2\pi r}=-(\partial_r\psi_{i,A}^{(k)})^2-V^{(k)}(r)(\psi_{i,A}^{(k)})^2\ , \qquad (\text{A.28})
$$

$$
\frac{\tilde{\mathcal{L}}^{(k;0)}}{2\pi r}=-(\partial_r\tilde{\psi}_{0,A}^{(0)})^2-\tilde{V}_0^{(k)}(r)(\tilde{\psi}_{0,A}^{(k)})^2\ ,\qquad \frac{\tilde{\mathcal{L}}^{(k;i)}}{2\pi r}=-(\partial_r\tilde{\psi}_{i,A}^{(0)})^2-\tilde{V}^{(k)}(r)(\tilde{\psi}_{i,A}^{(k)})^2\ , \qquad (\text{A.29})
$$

where A denotes the flavor index: $A = 1, 2, \cdots, N_{\rm f} - N_{\rm c}$ and $i = 1, \cdots, N_{\rm c} - 1$. The Schrödinger potentials are defined as

$$V_0^{(k)} = \frac{(N_{\rm c}-1)g^2}{N_{\rm c}} B + \frac{e^2}{2} A + \frac{2(\lambda_0 + (N_{\rm c}-1)\lambda_1)^2}{N_{\rm c}^2} + \left(\frac{f_0 + (N_{\rm c}-1) f_1 + N_{\rm c}(k-1)}{r N_{\rm c}} \right)^2 , \quad \text{(A.30)}$$

$$V^{(k)} = -\frac{g^2}{N_{\rm c}} B + \frac{e^2}{2} A + \frac{2(\lambda_0 - \lambda_1)^2}{N_{\rm c}^2} + \left(\frac{f_0 - f_1 + N_{\rm c} k}{r N_{\rm c}} \right)^2 , \quad \text{(A.31)}$$

$$\tilde{V}_0^{(k)} = -\frac{(N_{\rm c}-1)g^2}{N_{\rm c}} B - \frac{e^2}{2} A + \frac{2(\lambda_0 + (N_{\rm c}-1)\lambda_1)^2}{N_{\rm c}^2} + \left(\frac{f_0 + (N_{\rm c}-1) f_1 + N_{\rm c}(k-1)}{r N_{\rm c}} \right)^2 , \quad \text{(A.32)}$$

$$\tilde{V}^{(k)} = \frac{g^2}{N_{\rm c}} B - \frac{e^2}{2} A + \frac{2(\lambda_0 - \lambda_1)^2}{N_{\rm c}^2} + \left(\frac{f_0 - f_1 + N_{\rm c} k}{r N_{\rm c}} \right)^2 . \quad \text{(A.33)}$$

As expected, we have obtained only four different Schrödinger-type equations independent of $N_{\rm c}, N_{\rm f}$ ($N_{\rm c} > 1, N_{\rm f} > N_{\rm c}$). Therefore, the results we found in the minimal example with $N_{\rm c} = 2, N_{\rm f} = 3$ in the body of this paper are valid more generally. Also, when the gauge couplings are fine-tuned as $e^2/2 = g^2/N_{\rm c}$, the $N_{\rm c}$-dependence disappears from the equations altogether.

References

[1] G. 't Hooft, Nucl. Phys. B **190**, 455 (1981); S. Mandelstam, Phys. Lett. B **53**, 476 (1975).

[2] A. Hanany and D. Tong, JHEP **0307** (2003) 037 [arXiv:hep-th/0306150].

[3] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673** (2003) 187 [arXiv:hep-th/0307287].

[4] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, Nucl. Phys. B **686** (2004) 119 [arXiv:hep-th/0312233].

[5] D. Tong, Phys. Rev. D **69** (2004) 065003 [arXiv:hep-th/0307302].

[6] M. Shifman and A. Yung, Phys. Rev. D **70** (2004) 045004 [arXiv:hep-th/0403149].

[7] A. Hanany and D. Tong, JHEP **0404** (2004) 066 [arXiv:hep-th/0403158].

[8] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71** (2005) 045010 [arXiv:hep-th/0412082].

[9] M. Eto, L. Ferretti, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Nucl.Phys. **B780** 161-187, 2007 [arXiv: hep-th/0611313].

[10] J. M. Baptista, arXiv:0810.3220 [hep-th].

[11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39** (2006) R315 [arXiv:hep-th/0602170].

[12] M. Shifman and A. Yung, Rev. Mod. Phys. **79** (2007) 1139 [arXiv:hep-th/0703267].

[13] D. Tong, arXiv:hep-th/0509216.

[14] D. Tong, arXiv:0809.5060 [hep-th].

[15] K. Hashimoto and D. Tong, JCAP **0509** (2005) 004 [arXiv:hep-th/0506022].

[16] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. **96** (2006) 161601 [arXiv:hep-th/0511088].

[17] R. Auzzi, M. Shifman and A. Yung, Phys. Rev. D **73** (2006) 105012 [Erratum-ibid. D **76** (2007) 109901] [arXiv:hep-th/0511150].

[18] M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, Phys. Rev. D **74** (2006) 065021 [arXiv:hep-th/0607070].

[19] M. Shifman and A. Yung, Phys. Rev. D **73** (2006) 125012 [arXiv:hep-th/0603134].

[20] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Phys. Rev. D **76** (2007) 105002 [arXiv:0704.2218 [hep-th]].

[21] T. Vachaspati and A. Achucarro, Phys. Rev. D **44**, 3067 (1991).

[22] A. Achucarro and T. Vachaspati, Phys. Rept. **327** (2000) 347 [Phys. Rept. **327** (2000) 427] [arXiv:hep-ph/9904229].

[23] R. Auzzi, M. Eto and W. Vinci, JHEP **0711** (2007) 090 [arXiv:0709.1910 [hep-th]].

[24] R. Auzzi, M. Eto and W. Vinci, JHEP **0802** (2008) 100 [arXiv:0711.0116 [hep-th]]. M. Eto, arXiv:0810.4895 [hep-th].

[25] L. Ferretti and K. Konishi, arXiv:hep-th/0602252.

[26] L. Ferretti, S. B. Gudnason and K. Konishi, Nucl. Phys. B **789** (2008) 84 [arXiv:0706.3854 [hep-th]].

[27] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, arXiv:0802.1020 [hep-th]. W. Vinci, arXiv:0810.2449 [hep-th].

[28] K. Konishi, arXiv:0809.1374 [hep-th].

[29] D. Dorigoni, K. Konishi and K. Ohashi, arXiv:0801.3284 [hep-th].

[30] M. Edalati and D. Tong, JHEP **0705** (2007) 005 [arXiv:hep-th/0703045]; D. Tong, JHEP **0709** (2007) 022 [arXiv:hep-th/0703235]; M. Shifman and A. Yung, Phys. Rev. D **77** (2008) 125016 [arXiv:0803.0158 [hep-th]]; M. Shifman and A. Yung, Phys. Rev. D **77** (2008) 125017 [arXiv:0803.0698 [hep-th]];

[31] N. Seiberg, Phys. Rev. D **49**, 6857 (1994) [arXiv:hep-th/9402044].

[32] M. Shifman and A. Yung, Phys. Rev. D **76** (2007) 045005 [arXiv:0705.3811 [hep-th]]; S. Bolognesi, arXiv:0807.2456 [hep-th].

[33] V. Markov, A. Marshakov and A. Yung, Nucl. Phys. B **709** (2005) 267 [arXiv:hep-th/0408235].

[34] R. Auzzi and S. P. Kumar, arXiv:0810.3201 [hep-th].

[35] J. Polchinski and M. J. Strassler, arXiv:hep-th/0003136.

[36] G. 't Hooft, Nucl. Phys. B **153** (1979) 141.

[37] A. A. Penin, V. A. Rubakov, P. G. Tinyakov and S. V. Troitsky, Phys. Lett. B **389** (1996) 13 [arXiv:hep-ph/9609257].

[38] P. Goddard, J. Nuyts and D. I. Olive, Nucl. Phys. B **125**, 1 (1977); F. A. Bais, Phys. Rev. D **18**, 1206 (1978); E. J. Weinberg, Nucl. Phys. B **167**, 500 (1980).

[39] M. Eto, K. Hashimoto and S. Terashima, JHEP **0709**, 036 (2007) [arXiv:0706.2005 [hep-th]].

[40] M. Hindmarsh, Phys. Rev. Lett. **68** (1992) 1263; Nucl. Phys. B **392** (1993) 461 [arXiv:hep-ph/9206229].

[41] A. Achucarro, K. Kuijken, L. Perivolaropoulos and T. Vachaspati, Nucl. Phys. B **388** (1992) 435.

[42] R. A. Leese, Phys. Rev. D **46** (1992) 4677; R. A. Leese and T. M. Samols, Nucl. Phys. B **396** (1993) 639.

[43] A. A. Abrikosov, Sov. Phys. JETP **5** (1957) 1174 [Zh. Eksp. Teor. Fiz. **32** (1957) 1442].

[44] H. B. Nielsen and P. Olesen, Nucl. Phys. B **61** (1973) 45.

[45] G. Carlino, K. Konishi and H. Murayama, Nucl. Phys. B **590** (2000) 37 [arXiv:hep-th/0005076].

[46] U. Lindström and M. Roček, Nucl. Phys. B **222** (1983) 285; I. Antoniadis and B. Pioline, Int. J. Mod. Phys. A **12**, 4907 (1997) [arXiv:hep-th/9607058].

[47] A. I. Vainshtein and A. Yung, Nucl. Phys. B **614**, 3 (2001) [arXiv:hep-th/0012250].

[48] N. S. Manton, Phys. Lett. B **110**, 54 (1982).

[49] E. J. Weinberg, Nucl. Phys. B **167**, 500 (1980).